

On Orevkov's rational cuspidal plane curves

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Abstract

In this note, we consider rational cuspidal plane curves having exactly one cusp whose complements have logarithmic Kodaira dimension two. We classify such curves with the property that the strict transforms of them via the minimal embedded resolution of the cusp have the maximal self-intersection number. We show that the curves given by the classification coincide with those constructed by Orevkov.

1 Introduction

Let C be a curve on $\mathbf{P}^2 = \mathbf{P}^2(\mathbf{C})$. A singular point of C is said to be a *cusp* if it is a locally irreducible singular point. We say that C is *cuspidal* (resp. *unicuspidal*) if C has only cusps (resp. one cusp) as its singular points. We denote by $\bar{\kappa} = \bar{\kappa}(\mathbf{P}^2 \setminus C)$ the logarithmic Kodaira dimension of the complement $\mathbf{P}^2 \setminus C$. Let C' denote the strict transform of a rational unicuspidal plane curve C via the minimal embedded resolution of the cusp of C . By [Y], $\bar{\kappa} = -\infty$ if and only if $(C')^2 > -2$. By [Ts, Proposition 2], there exist no rational cuspidal plane curves with $\bar{\kappa} = 0$. See also [K1, O]. Thus $\bar{\kappa} \geq 1$ if and only if $(C')^2 \leq -2$. In [To], rational unicuspidal plane curves with $\bar{\kappa} = 1$ have already been classified. It was Orevkov [O] who constructed two sequences C_{4k}, C_{4k}^* ($k = 1, 2, \dots$) of rational unicuspidal plane curves with $\bar{\kappa} = 2$. See Section 3 for details. The purpose of this note is to classify rational unicuspidal plane curves C with $\bar{\kappa} = 2$ and $(C')^2 = -2$. The main result of this note is the following:

THEOREM 1. *Let C be a rational unicuspidal plane curve with $\bar{\kappa} = 2$. Then C is projectively equivalent to one of the Orevkov's curves if and only if $(C')^2 = -2$.*

For a plane curve C , we denote by $\bar{P}_m(\mathbf{P}^2 \setminus C)$ the logarithmic m -genus of the complement $\mathbf{P}^2 \setminus C$. In [K2], the curve C_4 was characterized by $\bar{\kappa}$ and \bar{P}_4 . The following theorem characterizes C_4 and C_4^* by $\bar{\kappa}$, \bar{P}_2 and \bar{P}_3 .

THEOREM 2. *A reduced plane curve C is projectively equivalent to C_4 or C_4^* if and only if $\bar{\kappa}(\mathbf{P}^2 \setminus C) \geq 0$ and $\bar{P}_2(\mathbf{P}^2 \setminus C) = \bar{P}_3(\mathbf{P}^2 \setminus C) = 0$.*

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2 Preliminaries

In this section, we prepare for the proof of our theorems.

2.1 Linear chains

Let D be a divisor on a smooth surface V , $\varphi : V' \rightarrow V$ a composite of successive blowings-up and $B \subset V'$ a divisor. We say that φ *contracts* B to D , or simply that B *shrinks to* D if $\varphi(\text{Supp } B) = \text{Supp } D$ and each center of blowings-up of φ is on D or one of its preimages. Let D_1, \dots, D_r be the irreducible components of D . We call D an *SNC-divisor* if D is a reduced effective divisor, each D_i is smooth, $D_i D_j \leq 1$ for distinct D_i, D_j , and $D_i \cap D_j \cap D_k = \emptyset$ for distinct D_i, D_j, D_k . Assume that D is an SNC-divisor and that each D_i is projective. Let $\Gamma = \Gamma(D)$ denote the dual graph of D . We give the vertex corresponding to a component D_i the weight D_i^2 . We sometimes do not distinguish between D and its weighted dual graph Γ . We use the following notation and terminology (cf. [F, Section 3] and [MT1, Chapter 1]). A blowing-up at a point $P \in D$ is said to be *sprouting* (resp. *subdivisional*) *with respect to* D if P is a smooth point (resp. node) of D . A component D_i is called a *branching component* of D if $D_i(D - D_i) \geq 3$.

Assume that Γ is connected and linear. In cases where $r > 1$, the weighted linear graph Γ together with a direction from an endpoint to the other is called a *linear chain*. By definition, the empty graph \emptyset and a weighted graph consisting of a single vertex without edges are linear chains. If necessary, renumber D_1, \dots, D_r so that the direction of the linear chain Γ is from D_1 to D_r and $D_i D_{i+1} = 1$ for $i = 1, \dots, r-1$. We denote Γ by $[-D_1^2, \dots, -D_r^2]$. We sometimes write Γ as $[D_1, \dots, D_r]$. The linear chain is called *rational* if every D_i is rational. In this note, we always assume that every linear chain is rational. The linear chain Γ is called *admissible* if it is not empty and $D_i^2 \leq -2$ for each i . Set $r(\Gamma) = r$. We define the *discriminant* $d(\Gamma)$ of Γ as the determinant of the $r \times r$ matrix $(-D_i D_j)$. We set $d(\emptyset) = 1$.

Let $A = [a_1, \dots, a_r]$ be a linear chain. We use the following notation if $A \neq \emptyset$:

$${}^t A := [a_r, \dots, a_1], \quad \overline{A} := [a_2, \dots, a_r], \quad \underline{A} := [a_1, \dots, a_{r-1}].$$

The discriminant $d(A)$ has the following properties ([F, Lemma 3.6]).

LEMMA 3. *Let $A = [a_1, \dots, a_r]$ be a linear chain.*

- (i) *If $r > 1$, then $d(A) = a_1 d(\overline{A}) - d(\overline{\overline{A}}) = d({}^t A) = a_r d(\underline{A}) - d(\underline{\underline{A}})$.*
- (ii) *If $r > 1$, then $d(\overline{A}) d(\underline{A}) - d(A) d(\overline{\underline{A}}) = 1$.*
- (iii) *If A is admissible, then $\gcd(d(A), d(\overline{A})) = 1$ and $d(A) > d(\overline{A}) > 0$.*

Let $A = [a_1, \dots, a_r]$ be an admissible linear chain. The rational number $e(A) := d(\overline{A})/d(A)$ is called the *inductance* of A . By [F, Corollary 3.8], the function e defines a one-to-one correspondence between the set of all the admissible linear chains and the set of rational numbers in the interval $(0, 1)$. For a given admissible linear chain A , the admissible linear chain $A^* := e^{-1}(1 - e({}^t A))$ is called the *adjoint* of A ([F, 3.9]). Admissible linear chains and their adjoints have the following properties ([F, Corollary 3.7, Proposition 4.7]).

LEMMA 4. *Let A and B be admissible linear chains.*

- (i) *If $e(A) + e(B) = 1$, then $d(A) = d(B)$ and $e({}^t A) + e({}^t B) = 1$.*
- (ii) *We have $A^{**} = A$, ${}^t(A^*) = ({}^t A)^*$ and $d(A) = d(A^*) = d(\overline{A^*}) + d(\underline{A})$.*
- (iii) *The linear chain $[A, 1, B]$ shrinks to $[0]$ if and only if $A = B^*$.*

For integers a, n with $n \geq 0$, we define $[(a)_n] = [\overbrace{a, \dots, a}^n]$, $t_n = [2_n]$. For non-empty linear chains $A = [a_1, \dots, a_r]$, $B = [b_1, \dots, b_s]$, we write $A * B = [\underline{A}, a_r + b_1 - 1, \overline{B}]$, $A^{*n} = \overbrace{A * \dots * A}^n$, where $n \geq 1$. We remark that $(A * B) * C = A * (B * C)$ for non-empty linear chains A, B and C . By using Lemma 3 and Lemma 4, we can show the following lemma.

LEMMA 5. *Let $A = [a_1, \dots, a_r]$ be an admissible linear chain.*

- (i) *For a positive integer n , we have $[A, n + 1]^* = t_n * A^*$.*
- (ii) *We have $A^* = t_{a_r-1} * \dots * t_{a_1-1}$.*
- (iii) *If there exist positive integers m, n such that $[A, m + 1] = [n + 1, A]$ (resp. $A * t_m = t_n * A$), then $m = n$, $a_1 = \dots = a_r = n + 1$ (resp. $A = t_n^{*r(A^*)}$).*

The following two lemmas describe the processes of contractions of special linear chains. The first one can be proved easily. We prove the second one.

LEMMA 6. *Let A be an admissible linear chain and B a non-empty linear chain. Suppose that a composite π of blowings-down contracts $[A, 1]$ to B .*

- (i) *The linear chain B is the image of the first $r(B)$ curves of A . We have $A = B * t_n$, where $n = r(A) + 1 - r(B)$.*
- (ii) *Every blowing-up of π is sprouting with respect to B or its preimage.*
- (iii) *The exceptional curve of each blowing-up of π is a unique (-1) -curve in the preimage of B .*

Conversely, $[B * t_n, 1]$ shrinks to B for a given positive integer n and a non-empty linear chain B .

LEMMA 7. Let A, B be admissible linear chains and c a positive integer. Suppose that a composite π of blowings-down contracts $[A, 1, B]$ to $[c, 1]$.

- (i) The first curve of $[c, 1]$ is the image of the first curve of A . We have $n := r(A) - r(B^*) \geq 0$ and $A = [c, t_n] * B^*$. In particular, $n = 0$ if $c = 1$.
- (ii) The first n blowings-up of π are sprouting and the remaining ones are subdivisational with respect to $[c, 1]$ or its preimages. The composite of the subdivisational blowings-up contracts $[A, 1, B]$ to $[c, t_n, 1]$.
- (iii) The exceptional curve of each blowing-up of π is a unique (-1) -curve in the preimage of $[c, 1]$.

PROOF. Write $A = [a_1, \dots, a_r]$, $B = [b_1, \dots, b_s]$. We prove the assertions by induction on $r + s \geq 2$. After the first blowing-down of π , $[A, 1, B]$ becomes $T := [\underline{A}, a_r - 1, b_1 - 1, \overline{B}]$. The last blowing-up of π satisfies (iii) and is subdivisational with respect to T . Suppose $r + s = 2$. We have $T = [c, 1]$, $\underline{A} = \overline{B} = \emptyset$, $b_1 = 2$ and $c = a_r - 1$. By Lemma 5, we obtain $B^* = [2]$ and $n = 0$. Hence $A = [c] * t_1 = [c, t_n] * B^*$. The remaining assertions are clear in this case. Assume $r + s \geq 3$. We have $T \neq [c, 1]$. Since A and B are admissible, a_r or b_1 must be equal to 2. If $a_r = b_1 = 2$, then $T = [\underline{A}, 1, 1, \overline{B}]$, which is contracted to $[\dots, 0, \dots]$ by the second blowing-down. But the latter linear chain cannot shrink to $[c, 1]$. Hence either a_r or b_1 must be greater than 2.

Case (1): $a_r = 2$, $b_1 > 2$. If $r = 1$, then $[b_s, \dots, b_2, b_1 - 1, 1]$ shrinks to $[1, c]$. By Lemma 6, $[b_s, \dots, b_2, b_1 - 1] = [1, c] * t_{s-1}$. Thus $b_s = 1$, which is a contradiction. Hence $r > 1$. Since \underline{A} is admissible, we have $\underline{A} = [c, t_{n'}] * [b_1 - 1, \overline{B}]^*$ by the induction hypothesis, where $n' = r - r([b_1 - 1, \overline{B}]^*) - 1$. Hence $A = [c, t_{n'}] * [[b_1 - 1, \overline{B}]^*, 2]$. By Lemma 5, we obtain $[[b_1 - 1, \overline{B}]^*, 2] = (t_1 * [b_1 - 1, \overline{B}])^* = B^*$ and $r([b_1 - 1, \overline{B}]^*) = r(B^*) - 1$. The remaining assertions follow from the induction hypothesis.

Case (2): $a_r > 2$, $b_1 = 2$. If $s = 1$, then $[\underline{A}, a_r - 1, 1]$ shrinks to $[c, 1]$. By Lemma 6, $[\underline{A}, a_r - 1] = [c, 1] * t_{r-1} = [c, t_{r-1}]$. Hence $A = [c, t_{r-1}] * t_1 = [c, t_{r-1}] * B^*$. The remaining assertions also follow from Lemma 6 in this case. If $s > 1$, then we have $[\underline{A}, a_r - 1] = [c, t_{n'}] * (\overline{B})^*$ by the induction hypothesis, where $n' = r - r((\overline{B})^*)$. By Lemma 5, we obtain $A = [c, t_{n'}] * (\overline{B})^* * t_1 = [c, t_{n'}] * [2, \overline{B}]^* = [c, t_{n'}] * B^*$ and $r((\overline{B})^*) = r(B^*)$. The remaining assertions follow from the induction hypothesis. \square

The following corollary to Lemma 7 describes the process of the contractions of linear chains in Lemma 4 (iii).

COROLLARY 8. *Let A and B be admissible linear chains. Suppose that a composite π of blowings-down contracts $[A, 1, B]$ to $[0]$.*

- (i) *The first blowing-up of π is sprouting with respect to $[0]$ and the remaining ones are subdivisional with respect to preimages of $[0]$.*
- (ii) *The exceptional curve of each blowing-up of π except the first one is a unique (-1) -curve in the preimage of $[0]$.*

The next one is a corollary to Lemma 4 (iii), Lemma 6 and Lemma 7. It will be used to describe the process of the resolutions of cusps.

COROLLARY 9. *Let a be a positive integer and A an admissible linear chain. Let B be a linear chain which is empty or admissible. Assume that a composite π of blowings-down contracts $[A, 1, B]$ to $[a]$ and that $[a]$ is the image of A under π .*

- (i) *The linear chain $[a]$ is the image of the first curve of A . There exists a positive integer n such that $A^* = [B, n + 1, t_{a-1}]$. Moreover, $A = [a] * t_n * B^*$ if $B \neq \emptyset$.*
- (ii) *The first n blowings-up of π are sprouting and the remaining ones are subdivisional with respect to $[a]$ and its preimages. The composite of the subdivisional blowings-up contracts $[A, 1, B]$ to $[[a] * t_n, 1]$.*
- (iii) *The exceptional curve of each blowing-up of π is a unique (-1) -curve in the preimage of $[a]$.*

Conversely, $[[a] * t_n * B^*, 1, B]$ shrinks to $[a]$ for given positive integers a, n and an admissible linear chain B .

The following corollary follows from Corollary 9 (ii).

COROLLARY 10. *Let the notation and the assumption be as in Corollary 9 and b an integer. Then π contracts $[A, 1, B, b]$ to $[a, b - n]$. The second curve of $[a, b - n]$ is the image of the last curve of $[A, 1, B, b]$.*

2.2 Resolution of a cusp

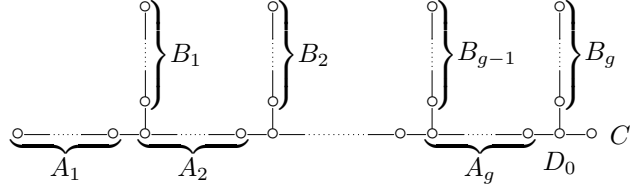
Let C be a curve on a smooth surface V . Suppose that C has a cusp P . Let $\sigma : V' \rightarrow V$ be the minimal embedded resolution of the cusp. That is, σ is the composite of the shortest sequence of blowings-up such that the strict transform C' of C intersects $\sigma^{-1}(P)$ transversally. Let $V' = V_n \xrightarrow{\sigma_{n-1}} V_{n-1} \rightarrow \cdots \rightarrow V_2 \xrightarrow{\sigma_1} V_1 \xrightarrow{\sigma_0} V_0 = V$ be the blowings-up of σ . The following lemma follows from the assumptions that P is a cusp and σ is minimal.

LEMMA 11. For $i \geq 1$, the strict transform of C on V_i intersects $(\sigma_0 \circ \cdots \circ \sigma_{i-1})^{-1}(P)$ in one point, which is on the exceptional curve of σ_{i-1} . The point of intersection is the center of σ_i if $i < n$.

We prove the following lemma.

LEMMA 12. The following assertions hold (cf. [BK, MaSa]).

- (i) The dual graph of $\sigma^{-1}(C)$ has the following shape, where $g \geq 1$, D_0 is the exceptional curve of σ_{n-1} and A_1 contains the exceptional curve of σ_0 by definition.



We number the irreducible components $A_{i,j}$ of A_i (resp. $B_{i,j}$ of B_i) from the left-hand side to the right (resp. the bottom to the top) in the above figure.

- (ii) The morphism σ can be written as $\sigma = \sigma_0 \circ \rho'_1 \circ \rho''_1 \circ \cdots \circ \rho'_g \circ \rho''_g$, where each ρ'_i (resp. ρ''_i) consists of sprouting (resp. subdivisional) blowings-up of σ with respect to preimages of P .
- (iii) The morphisms $\rho_i := \rho'_i \circ \rho''_i$ have the following properties.

- (a) Each ρ_i maps A_i to a (-1) -curve, which is the image of $A_{i,1}$.
- (b) ρ_g contracts $A_g + D_0 + B_g$ to $A_{g,1}$ and ρ_i contracts $A_i + A_{i+1,1} + B_i$ to $A_{i,1}$ for $i < g$.

PROOF. For the sake of simplicity, we do not distinguish between a curve and its strict transforms via blowings-up. The second blowing-up of σ is sprouting with respect to the exceptional curve of σ_0 . Since P is a cusp and σ is minimal, the last blowing-up of σ must be subdivisional with respect to the preimage of P . These facts show the assertion (ii). Let $E_{0,0}$ denote the exceptional curve of σ_0 and $E_{i,0}$ the exceptional curve of the last blowing-up of ρ''_i for each i . Put $E_0 = \emptyset$. Let E_i denote the exceptional curve of ρ_i . By Lemma 11, we infer that the dual graph of the sum of $E_{i-1,0}$ and the exceptional curve of ρ'_i is linear. Hence the dual graph of $E_{i-1,0} + E_i$ is linear. It follows that $E_{1,0}, \dots, E_{g-1,0}, E_{g,0} = D_0$ are all the branching components of $\sigma^{-1}(C)$. The divisor $E_{i-1,0} + E_i - E_{i,0}$ consists of two connected components. Let A_i denote the one containing $E_{i-1,0}$ and B_i the remaining one. Then A_i , B_i and ρ_i have the desired properties. \square

We give the weighted graphs A_1, \dots, A_g (resp. B_1, \dots, B_g) the direction from the left-hand side to the right (resp. from the bottom to the top) of the figure in Lemma 12. With these directions, we regard A_i and B_i as linear chains. By Lemma 11, these linear chains are admissible. Let o_i denote the number of the blowings-up in ρ'_i . The following proposition follows from Corollary 9.

PROPOSITION 13. *The following assertions hold for $i = 1, \dots, g$.*

- (i) *We have $A_i = t_{o_i} * B_i^*$, $A_i^* = [B_i, o_i + 1]$.*
- (ii) *The (-1) -curve $A_{i,1}$ is the image of the first curve of A_i under ρ_i .*
- (iii) *The linear chain A_i contains an irreducible component E with $E^2 \leq -3$.*

We will use the next lemma to prove some properties of the Orevkov's curves.

LEMMA 14. *Let D' be an SNC-divisor on a smooth surface V' . Suppose the following conditions are satisfied.*

- (i) *The weighted dual graph of D' consists of a (-1) -curve D_0 and admissible rational linear chains $A_1, B_1, \dots, A_g, B_g$, $g \geq 1$. They meet each other in the way described in Lemma 12 (i).*
- (ii) *For $i = 1, \dots, g$, there exists a positive integer o_i such that $A_i = t_{o_i} * B_i^*$, or equivalently $A_i^* = [B_i, o_i + 1]$.*

Then the following assertions hold.

- (a) *The divisor D' shrinks to a point P by blowings-down $\sigma : V' \rightarrow V$. The way of blowings-down to contract D' to a point is unique.*
- (b) *Let C' be a smooth curve on V' . If C' intersects only D_0 at one point transversally among the irreducible components of D' , then $\sigma(C')$ is smooth outside of P and has a cusp at P , whose minimal embedded resolution coincides with σ .*

PROOF. (a) By Corollary 9, $[A_g, D_0, B_g]$ shrinks to a (-1) -curve, which is the image of the first curve $A_{g,1}$ of A_g . The image of $[A_{g-1}, A_{g,1}, B_{g-1}]$ under the above contraction shrinks to a (-1) -curve, which is the image of the first curve of A_{g-1} . Continuing in this way, we get blowings-down $\sigma : V' \rightarrow V$ which contracts D' to a point P . The uniqueness follows from Corollary 9 (iii).

(b) Since C' is smooth, $\sigma(C')$ is also smooth outside of P . If the center of a blowing-up of σ is not on the image of C' , then those of the remaining blowings-up are not on the images of C' by Corollary 9 (iii). This contradicts the assumption that C' intersects D_0 . Hence the center of each blowing-up of σ is on the image of C' . The remaining assertions of (b) follow from this fact. \square

3 Orevkov's curves and proof of the “only if” part of Theorem 1

In this section, we prove some properties of Orevkov's curves, from which the “only if” part of Theorem 1 follows. In [O], Orevkov constructed two sequences C_{4k}, C_{4k}^* ($k = 1, 2, \dots$) of rational unicuspidal plane curves with $\bar{\kappa} = 2$ in the following way. Let N be a nodal cubic. Let Γ_1, Γ_2 denote the two analytic branches of N at the node. Let $\phi : W \rightarrow \mathbf{P}^2$ denote the composite of 7-times of blowings-up such that the center of the first one is the node and every center of the remaining ones is the point of intersection of the strict transform of Γ_1 and the exceptional curve of the previous blowing-up. The dual graph of the exceptional curve E of ϕ is connected and linear. The curve E consists of 6-pieces of (-2) -curves and one (-1) -curve E' as an endpoint and intersects the strict transform of N at its two endpoints.

Let $\phi' : W \rightarrow \mathbf{P}^2$ denote the contraction of the strict transform of N and the 6-pieces of (-2) -curves in E . Put $f = \phi' \circ \phi^{-1}$. The curve $\phi'(E')$ is a nodal cubic. Let Γ denote one of the two analytic branches of $\phi'(E')$ at the node such that the center of the second blowing-up of ϕ' is not on its strict transform. We may assume $\phi'(E') = N$ and $\Gamma = \Gamma_1$ by composing a suitable projective transformation to f . Let C_0 be the tangent line at a flex of N and C_0^* an irreducible conic meeting with N only at one smooth point. See [O, AT] or the appendix for the existence of C_0^* . Orevkov defined C_{4k}, C_{4k}^* as $C_{4k} = f(C_{4k-4}), C_{4k}^* = f(C_{4k-4}^*)$ ($k = 1, 2, \dots$). They have a cusp at the node and tangent to Γ_2 at the node.

LEMMA 15. *Let C be a rational unicuspidal plane curve, $\sigma : V \rightarrow \mathbf{P}^2$ the minimal embedded resolution of the cusp and C' the strict transform of C via σ . Put $D = \sigma^{-1}(C)$. Let $A_1, B_1, \dots, A_g, B_g, D_0$ denote the linear chains given for the cusp by Lemma 12.*

(i) *The curve C can be constructed in the same way as C_4 (resp. C_4^*) if and only if C satisfies the following conditions.*

- (a) $g = 1, A_1 = [t_6, 4], B_1 = t_2$ (resp. $A_1 = [t_6, 7], B_1 = t_5$).
- (b) *There exists a (-1) -curve E_0 such that it meets with D at two points transversally and intersects only the first curve and the last curve of A_1 among the irreducible components of D .*

(ii) *The curve C can be constructed in the same way as C_{4k+4} (resp. C_{4k+4}^*) for some $k \geq 1$ if and only if C satisfies the following conditions.*

- (a) $g = 2, A_1 = t_6^{*k+1}, B_1 = [7_k], A_2 = [4], B_2 = t_2$ (resp. $A_2 = [7], B_2 = t_5$).
- (b) *There exists a (-1) -curve E_0 such that it meets with D at two points transversally and intersects only the first curve of A_1 and the last curve of B_1 among the irreducible components of D .*

(iii) If C can be constructed in the same way as C_{4k} or C_{4k}^* for some $k \geq 1$, then $(C')^2 = -2$.

PROOF. The assertions for C_4 and C_4^* follow from their definition. We prove (ii) and (iii) for C_{4k+4} , $k \geq 1$. We can similarly deal with C_{4k+4}^* . We first show the “if” part of (ii) by induction on k . Let a_i and b_i denote the i -th curves of the linear chains A_1 and B_1 , respectively. For the sake of simplicity, we sometimes use the same symbols for the strict transforms them via a rational map which does not contract them.

Write σ as $\sigma = \sigma_2 \circ \sigma_1$, where σ_2 consists of seven blowings-up. By Corollary 9 (ii), the last six blowings-up of σ_2 are sprouting with respect to the preimages of the cusp. The weighted dual graph of the preimage of the cusp under σ_2 is the linear chain $[t_6, 1]$. By Corollary 9 (iii), the blowings-up of σ_1 are done over the point of intersection of t_6 and the (-1) -curve. From these facts, we see $[t_6, 1] = [\sigma_1(a_1), \dots, \sigma_1(a_6), \sigma_1(b_k)]$. The dual graph of $\sigma_1(E_0 + a_1 + \dots + a_6 + b_k)$ is a loop. We have $[1, t_6, 1] = [\sigma_1(E_0), \sigma_1(a_1), \dots, \sigma_1(a_6), \sigma_1(b_k)]$. Let $\varphi_1 : V_1 \rightarrow V_0$ denote the contraction of $\sigma_1(E_0 + a_1 + \dots + a_5)$ and $\varphi_0 : V_0 \rightarrow \mathbf{P}^2$ the contraction of $\varphi_1(\sigma_1(a_6))$. Put $\varphi = \varphi_0 \circ \varphi_1$.

We arrange the order of blowings-down of $\varphi \circ \sigma_1$ in the following way. We first perform six blowings-down $\varphi'_1 : V \rightarrow V'$ in the same way as φ_1 . It contracts $E_0 + a_1 + \dots + a_5$ to a point. Then we perform blowings-down $\sigma'_1 : V' \rightarrow V'_0$ in the same way as σ_1 . It contracts $\varphi'_1(D - (C' + a_1 + \dots + a_6 + b_k))$ to a point. Finally we perform the blowing-down $\varphi'_0 : V'_0 \rightarrow \mathbf{P}^2$ which contracts $\sigma'_1(\varphi'_1(a_6))$. The rational map $\varphi'_0 \circ \sigma'_1 \circ \varphi'_1 \circ (\varphi \circ \sigma_1)^{-1}$ is a projective transformation since it does not have exceptional curves. By Corollary 10, $\varphi'_1(a_6)$ (resp. $\varphi'_1(b_k)$) is a (-2) -curve (resp. (-1) -curve). The weighted dual graph of $D - (a_1 + \dots + a_6 + b_k)$ is unchanged by φ'_1 .

We decompose the exceptional curve $\varphi'_1(D - (C' + a_1 + \dots + a_5 + b_k))$ of $\varphi'_0 \circ \sigma'_1$ into linear chains $A'_1, B'_1, \dots, A'_{g'}, B'_{g'}, \varphi'_1(D_0)$. If $k = 1$, then we set $g' = 1$, $A'_1 = [\varphi'_1(a_6), \dots, \varphi'_1(a_{11}), \varphi'_1(A_2)]$ and $B'_1 = \varphi'_1(B_2)$. We have $(A'_1)^* = [t_6, 4]^* = [B'_1, 8]$. If $k > 1$, then we set $g' = 2$, $A'_1 = [\varphi'_1(a_6), \dots, \varphi'_1(a_{5k+6})]$, $B'_1 = [\varphi'_1(b_1), \dots, \varphi'_1(b_{k-1})]$, $A'_2 = \varphi'_1(A_2)$ and $B'_2 = \varphi'_1(B_2)$. We have $(A'_1)^* = [7_k] = [B'_1, 7]$. It follows from Lemma 14 that $\hat{C} := \varphi(\sigma_1(C'))$ is unicuspidal and that $\varphi'_0 \circ \sigma'_1$ is the minimal embedded resolution of the cusp. The linear chains $A'_1, B'_1, \dots, A'_{g'}, B'_{g'}$ coincide with those given for \hat{C} by Lemma 12. By the induction hypothesis ($k > 1$) and the assertion (i) ($k = 1$), \hat{C} can be constructed in the same way as C_{4k} . The curve $\varphi_1(\sigma_1(a_6))$ intersects $\varphi_1(\sigma_1(b_k))$ only at two points transversally. This shows that $\varphi(\sigma_1(b_k))$ is a nodal cubic. The morphism φ (resp. σ_2) performs blowings-up in the same way as ϕ (resp. ϕ'). Thus C can be constructed in the same way as C_{4k+4} .

We next show (iii) and the “only if” part of (ii). The curve C is the strict transform of an Orevkov’s curve C_{4k} via $f = \phi' \circ \phi^{-1}$. To avoid confusion,

we denote by N_i (resp. $\phi_i : W_i \rightarrow \mathbf{P}^2$, $\phi'_i : W_i \rightarrow \mathbf{P}^2$) the nodal cubic N (resp. the birational morphism ϕ, ϕ') which is used to make C_{4i+4} from C_{4i} for $i \leq k$. The curve C_{4k} is the strict transform of an Orevkov's curve C_{4k-4} via $f_{k-1} = \phi'_{k-1} \circ \phi_{k-1}^{-1}$. Let $\sigma : V \rightarrow \mathbf{P}^2$ denote the minimal embedded resolution of the cusp of C and e_i the exceptional curve of the i -th blowing-up of σ . We note that the strict transform of N_k via ϕ_k coincides with e_7 . Let $\sigma_k : V_k \rightarrow \mathbf{P}^2$ denote the minimal embedded resolution of the cusp of C_{4k} . From the definition of the Orevkov's curves, we infer that the centers of blowings-up of $\phi'_k : W_k \rightarrow \mathbf{P}^2$ (resp. $\phi'_{k-1} : W_{k-1} \rightarrow \mathbf{P}^2$) are the cusp of C (resp. C_{4k}) and its strict transforms. This shows that $\sigma : V \rightarrow \mathbf{P}^2$ (resp. $\sigma_k : V_k \rightarrow \mathbf{P}^2$) can be written as $\sigma = \phi'_k \circ \sigma'$ (resp. $\sigma_k = \phi'_{k-1} \circ \sigma'_k$), where σ' (resp. σ'_k) consists of blowings-up.

Let $A'_1, B'_1, \dots, A'_{g'}, B'_{g'}, D'_0$ denote the linear chains given by Lemma 12 for C_{4k} . If $k = 1$, they satisfy the conditions (a), (b) in (i). Otherwise they satisfy those in (ii) with k being replaced with $k - 1$ by the induction hypothesis. Let $\phi_{k,0} : W_{k,0} \rightarrow \mathbf{P}^2$ denote the first blowing-up of ϕ_k , which coincides with that of ϕ'_{k-1} . Let $\phi_{k,1}$ (resp. $\phi'_{k-1,1}$) denote the composite of the remaining blowings-up of ϕ_k (resp. ϕ'_{k-1}). Each blowing-up of $\phi_{k,1}$ is done over Γ_1 , while that of $\phi'_{k-1,1} \circ \sigma'_k$ is done over Γ_2 . This means that as a weighted graph, the strict transform of $A'_1 + B'_1 + \dots + A'_{g'} + B'_{g'} + D'_0$ on V via $\sigma_k^{-1} \circ \phi_k \circ \sigma' : V \rightarrow V_k$ is obtained by increasing the weight of the first curve of A'_1 by one, which is done by the first blowing-up of $\phi_{k,1}$. Moreover, $A_1 + B_1 + \dots + A_g + B_g + D_0$ is obtained by attaching the weighted dual graph of the strict transform of $e_1 + \dots + e_7$ on V to $\overline{A'_1} + B'_1 + A'_2 + B'_2 + \dots + A'_{g'} + B'_{g'} + D'_0$. The first curve of A'_1 is replaced with the strict transform of e_6 .

The curves e_6, e_7 and the strict transform of C on W_k intersect each other in the same way as $\phi_{k,1}(e_6), \phi_{k,1}(e_7)$ and the strict transform of C_{4k} on $W_{k,0}$ do. Furthermore, σ' performs blowings-up in the same way as $\phi'_{k-1,1} \circ \sigma'_k$. We have $(C')^2 = (C_{4k})^2 = -2$ by the induction hypothesis. The first blowing-up of σ' is done at $e_6 \cap e_7$ and each of the next five blowings-up of σ' is done at the point of intersection of the strict transform of e_7 and the exceptional curve of the previous blowing-up. Let $\sigma'' : W'_k \rightarrow W_k$ denote the composite of the first six blowings-up of σ' and e'_i, N'_{k+1} the strict transforms of e_i, N_{k+1} on W'_k , respectively. The dual graph of $e'_1 + \dots + e'_6 + e'_8 + \dots + e'_{13} + e'_7 + N'_{k+1}$ is a loop. We have $[e'_1, \dots, e'_6, e'_8, \dots, e'_{13}, e'_7, N'_{k+1}] = [t_5, 3, t_5, 1, 7, 1]$.

As we saw in the proof of the "if" part, if $k > 1$, then e'_{13} is the image of b'_k and $e'_6, e'_8, \dots, e'_{12}$ are those of a'_1, \dots, a'_6 , respectively, where a'_i (resp. b'_i) denotes the strict transform on V of the i -th curve of A'_1 (resp. B'_1) via $\sigma_k^{-1} \circ \phi_k \circ \sigma' : V \rightarrow V_k$. The remaining blowings-up of σ' are done over $e'_{12} \cap e'_{13}$. It follows from the definition of the Orevkov's curves that if $k = 1$, then $e'_6, e'_8, \dots, e'_{13}$ are the images of a'_1, \dots, a'_7 , respectively. The remaining blowings-up of σ' are done over a point on $e'_{13} \setminus (e'_1 + \dots + e'_{12})$. Let e''_i denote

the strict transform of e_i on V . If $k = 1$, then $g = 2$, $A_1 = [e'_1, \dots, e'_6, \overline{A'_1}] = t_6^{*2}$, $B_1 = [e''_7] = [7]$, $A_2 = [e''_{13}] = [a'_7] = [4]$ and $B_2 = B'_1$. If $k > 1$, then $g = 2$, $A_1 = [e'_1, \dots, e'_6, \overline{A'_1}] = t_6^{*k+1}$, $B_1 = [B'_1, e''_7] = [7_k]$, $A_2 = A'_2$ and $B_2 = B'_2$. The strict transform of N_{k+1} via σ satisfies the condition that E_0 must satisfy. \square

By Proposition 16 below, each C_{4k} (resp. C_{4k}^*) does not depend on the choice of N and C_0 (resp. C_0^*) up to the projective equivalence. The “only if” part of Theorem 1 follows from this fact and Lemma 15 (iii).

PROPOSITION 16. *Let $C^{(1)}$ and $C^{(2)}$ be plane curves. If there exists a positive integer k such that $C^{(1)}$ and $C^{(2)}$ can be constructed in the same way as C_{4k} , or they can be constructed in the same way as C_{4k}^* , then $C^{(1)}$ is projectively equivalent to $C^{(2)}$.*

PROOF. We only show the assertion for the case in which there exists $k \geq 2$ such that $C^{(1)}$ and $C^{(2)}$ can be constructed in the same way as C_{4k} . We can similarly deal with the remaining cases. For each i , let $\sigma^{(i)} : V^{(i)} \rightarrow \mathbf{P}^2$ denote the minimal embedded resolution of the cusp of $C^{(i)}$. Write $A_1, B_1, \dots, A_g, B_g, D_0$, etc. given by Lemma 12 for $C^{(i)}$ as $A_1^{(i)}, B_1^{(i)}, \dots, A_{g_i}^{(i)}, B_{g_i}^{(i)}, D_0^{(i)}$, etc. Let $E_0^{(i)}$ denote the (-1) -curve E_0 given for $C^{(i)}$ in Lemma 15 (ii). We define a birational morphism $\psi^{(i)} : V^{(i)} \rightarrow \mathbf{P}^2$ in the following way. It first contracts $D_0^{(i)} + B_2^{(i)}$ to a point. Then it contracts the image of $A_1^{(i)} + E_0^{(i)} + B_1^{(i)}$ to a point. The last blowing-down of $\psi^{(i)}$ contracts the image $a_1^{(i)}$ of the last curve of $A_1^{(i)}$ to a point. We infer that $a_1^{(i)}$ intersects the image of $A_2^{(i)}$ at two points transversally. It follows that $\psi^{(i)}(A_2^{(i)})$ is a nodal cubic and that $\psi^{(i)}(C^{(i)'})$ is the tangent line at a flex of $\psi^{(i)}(A_2^{(i)})$. We may assume that each nodal cubic $\psi^{(i)}(A_2^{(i)})$ is defined by the equation given in the appendix. We denote $\psi^{(i)}(A_2^{(i)})$ by N . Let O_1, O_2 and O_3 be the flexes of N defined in the appendix. There exists a positive integer $a \leq 3$ such that $\psi^{(1)}(C^{(1)'})$ is the tangent line at O_a . Furthermore, there exists a projective transformation h such that $h(N) = N$ and $h(\psi^{(1)}(C^{(1)'})) = \psi^{(2)}(C^{(2)'})$.

Let $\psi_j^{(i)} : V_j^{(i)} \rightarrow V_{j-1}^{(i)}$ denote the j -th blowing-up of $\psi^{(i)}$, where $V_0^{(i)} = \mathbf{P}^2$. Since h maps the center of $\psi_1^{(1)}$ to that of $\psi_1^{(2)}$, the rational map $h_1 = \psi_1^{(2)-1} \circ h \circ \psi_1^{(1)} : V_1^{(1)} \rightarrow V_1^{(2)}$ is an isomorphism. The center of $\psi_2^{(1)}$ is one of the two points of intersection of N and the exceptional curve of $\psi_1^{(1)}$. By replacing h with the composite of h and the projective transformation φ_a given in the appendix, if necessary, we may assume that h_1 maps the center of $\psi_2^{(1)}$ to that of $\psi_2^{(2)}$. Thus $\psi_2^{(2)-1} \circ h_1 \circ \psi_2^{(1)} : V_2^{(1)} \rightarrow V_2^{(2)}$ is an isomorphism. For the remaining blowings-up, there are no ambiguities in choices of centers. It follows that $h' = \psi^{(2)-1} \circ h \circ \psi^{(1)} : V^{(1)} \rightarrow V^{(2)}$ is an

isomorphism. Since h' maps the exceptional curve of $\sigma^{(1)}$ to that of $\sigma^{(2)}$, the rational map $\sigma^{(2)} \circ h' \circ \sigma^{(1)-1}$ is a projective transformation such that $\sigma^{(2)} \circ h' \circ \sigma^{(1)-1}(C^{(1)}) = C^{(2)}$. \square

4 Structure of \mathbf{C}^{**} -fibration

Let C be a rational unicuspidal plane curve and P the cusp of C . As in Section 2.2, let $\sigma : V \rightarrow \mathbf{P}^2$ denote the minimal embedded resolution of the cusp, σ_0 the first blowing-up of σ and C' the strict transform of C via σ . Put $D = \sigma^{-1}(C)$. Let D_0 denote the exceptional curve of the last blowing-up of σ . We decompose the dual graph of $\sigma^{-1}(P)$ into linear chains $A_1, B_1, \dots, A_g, B_g, D_0$ in the same way as in Section 2.2. By Lemma 12, there exists a decomposition $\sigma = \sigma_0 \circ \rho'_1 \circ \rho''_1 \circ \dots \circ \rho'_g \circ \rho''_g$, where each ρ'_i (resp. ρ''_i) consists of sprouting (resp. subdivisional) blowings-up with respect to preimages of P . Let o_i denote the number of the blowings-up in ρ'_i .

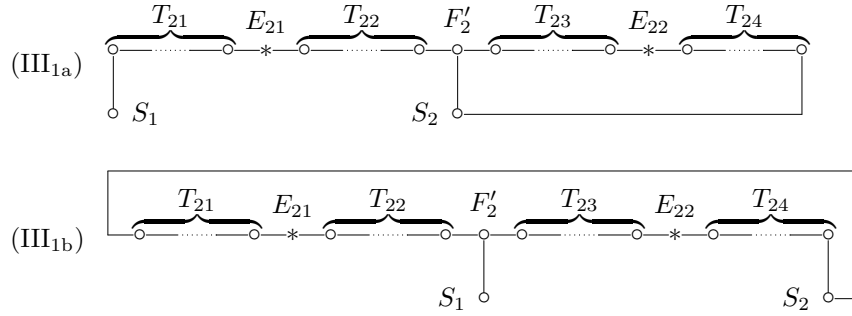
Assume that the rational unicuspidal plane curve C satisfies the conditions that $(C')^2 = -2$ and $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 2$. We see that one and only one of the two irreducible components of $D - D_0 - C'$ meeting with D_0 must be a (-2) -curve. Let F'_0 denote the (-2) -curve and S_2 the remaining one. Let $\varphi_0 : V \rightarrow V'$ be the contraction of D_0 and C' . Since $(F'_0)^2 = 0$ on V' , there exists a \mathbf{P}^1 -fibration $p' : V' \rightarrow \mathbf{P}^1$ such that F'_0 is a nonsingular fiber. Put $p = p' \circ \varphi_0 : V \rightarrow \mathbf{P}^1$. Since $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 2$, there exists an irreducible component S_1 of $D - D_0 - F'_0$ meeting with F'_0 on V . Put $F_0 = F'_0 + D_0 + C'$. The surface $X = V \setminus D$ is a \mathbf{Q} -homology plane. A general fiber of $p|_X$ is a curve $\mathbf{C}^{**} = \mathbf{P}^1 \setminus \{3 \text{ points}\}$. Such fibrations have already been classified in [MiSu]. We will use their result to prove our theorem.

There exists a birational morphism $\varphi : V \rightarrow \Sigma_n$ from V onto the Hirzebruch surface Σ_n of degree n for some n such that $p \circ \varphi^{-1} : \Sigma_n \rightarrow \mathbf{P}^1$ is a \mathbf{P}^1 -bundle. The morphism φ is the composite of the successive contractions of the (-1) -curves in the singular fibers of p . The curve S_1 (resp. S_2) is a 1-section (resp. 2-section) of p . The divisor D contains no other sections of p .

LEMMA 17. *We may assume that $\varphi(S_1 + S_2)$ is smooth. We have $\varphi(S_1)^2 = -1$ and $\varphi(S_2)^2 = 4$.*

PROOF. We only prove the first assertion. Suppose $\varphi(S_1 + S_2)$ has a singular point P . Let ϕ_1 be the blowing-up at P . Since $S_1 + S_2$ is smooth on V , we can choose the order of the blowings-up of φ such that $\varphi = \phi_1 \circ \varphi'$. Let F' be the strict transform via ϕ_1 of the fiber of $p \circ \varphi^{-1}$ passing through P . Let ϕ_2 be the contraction of F' . Since F' is an irreducible component of a singular fiber of $p \circ \varphi'^{-1}$, we can replace φ with $\phi_2 \circ \varphi'$. We infer that P can be resolved by repeating the above process. Hence we may assume that $\varphi(S_1 + S_2)$ is smooth. \square

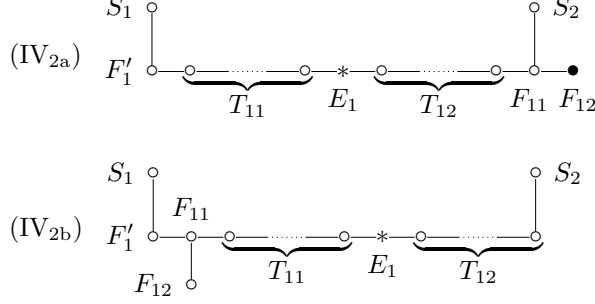
Each singular fiber of p intersects S_2 in at most two points. Suppose that there exists a singular fiber F_2 of p meeting with S_2 in two points. Let E_2 be the sum of the irreducible components of F_2 which are not components of D . Because D contains no loop, E_2 is not empty. Since $\bar{\kappa}(V \setminus D) = 2$, each irreducible component of E_2 meets with D in at least two points by [MT2, Main Theorem]. In [MiSu, Lemma 1.6], singular fibers of a \mathbf{C}^{**} -fibration with a 2-section were classified into several types. Among them, only singular fibers of type (I_1) and (III_1) satisfy the conditions that they meet with the 2-section in two points and that each irreducible component of E_2 meets with D in at least two points. From the fact that D contains no loop, we infer that F_2 is of type (III_1) . The dual graph of $F_2 + S_1 + S_2$ coincides with one of those in the following figure, where $*$ denotes a (-1) -curve and $E_2 = E_{21} + E_{22}$. The divisor $T_{2,i}$ may be empty for each i .



LEMMA 18. *We have $\varphi(F_2) = \varphi(F'_2)$, where F'_2 is the irreducible component of F_2 whose position in F_2 is illustrated in the above figure.*

PROOF. Suppose that φ contracts F'_2 . Write $\varphi = \phi_3 \circ \phi_2 \circ \phi_1$, where ϕ_2 is the contraction of F'_2 . If F_2 is of type (III_{1a}) , then $\phi_1(F'_2)\phi_1(S_1) = 0$ and $\phi_1(F'_2)\phi_1(S_2) = 1$ by Lemma 17. Since $\phi_1(F_2 - F'_2)\phi_1(F'_2) \geq 2$, we have $\phi_2(\phi_1(F_2))\phi_2(\phi_1(S_2)) \geq 3$, which is a contradiction. If F_2 is of type (III_{1b}) , then $\phi_1(F'_2)\phi_1(S_2) = 0$ by Lemma 17. We have $\phi_2(\phi_1(F_2))\phi_2(\phi_1(S_1)) \geq 2$, which is absurd. \square

Suppose that there exists a singular fiber F_1 of p which intersects S_2 in one point. Let E_1 be the sum of the irreducible components of F_1 which are not components of D . By the same reasoning as for F_2 , we deduce that F_1 is of type (IV_2) . See [MiSu, Lemma 1.6]. The dual graph of $F_1 + S_1 + S_2$ coincides with one of those in the following figure, where \bullet denotes a (-2) -curve. The divisor $T_{1,i}$ may be empty for each i .



We can choose the order of the blowings-down of φ such that $\varphi = \varphi' \circ \varphi_1 \circ \varphi''$, where φ_1 is the composite of all the contractions of irreducible components of F_1 .

LEMMA 19. *The morphism φ_1 contracts $\varphi''(T_{11} + E_1 + T_{12} + F_{11})$ to a (-1) -curve, which is the image of F_{11} , and then contracts the (-1) -curve and the image of F_{12} in this order. We have $\varphi(F_1) = \varphi(F'_1)$. Moreover, $(F'_1)^2 = F_{12}^2 = -2$ if F_1 is of type (IV_{2b}).*

PROOF. Suppose that F_1 is of type (IV_{2b}). Since $(F'_1)^2 \leq -2$, $F_{12}^2 \leq -2$, φ contracts F_{11} before the contractions of F'_1 and F_{12} . Since $\varphi(F_1)$ is smooth, $T_{11} + E_1 + T_{12}$ must be contracted to a point before the contraction of F_{11} . It follows that $(F'_1)^2 = F_{12}^2 = -2$. By Lemma 17, φ does not contract F'_1 .

Suppose that F_1 is of type (IV_{2a}). Assume φ contracts F'_1 . By Corollary 8, F'_1 is the exceptional curve of the first blowing-up of φ_1 . The remaining blowings-up are subdivisional with respect to the preimages of $\varphi_1(\varphi''(F_1))$. By Lemma 17, the center of the first blowing-up is not on $\varphi_1(\varphi''(S_2))$. This means that $F_1 S_2 = 2$, which is a contradiction. Thus φ does not contract F'_1 . By Corollary 8, F_{12} is the exceptional curve of the first blowing-up of φ_1 . Since the remaining blowings-up are subdivisional with respect to the preimages of $\varphi_1(\varphi''(F_1))$, we infer that the exceptional curve of the second blowing-up of φ_1 coincides with the image of F_{11} . \square

By the Riemann-Hurwitz formula, p has no more than two singular fibers which meet with S_2 in one point. By [MiSu, Lemma 2.3], p has one singular fiber of type (III₁). It follows that the dual graph of D must be one of those in Figure 1.

5 Proof of the “if” part of Theorem 1 and Theorem 2

Let the notation be as in the previous section. We determine which graphs in Figure 1 can be realized. With the direction from the left-hand side to the right of Figure 1, we regard T_{ij} ’s as linear chains. Put $s_i = -S_i^2$ and $f_i = -(F'_i)^2$ for each i . We have $s_2 \geq 3$, $s_1 \geq 2$ and $f_i \geq 2$ for each i .

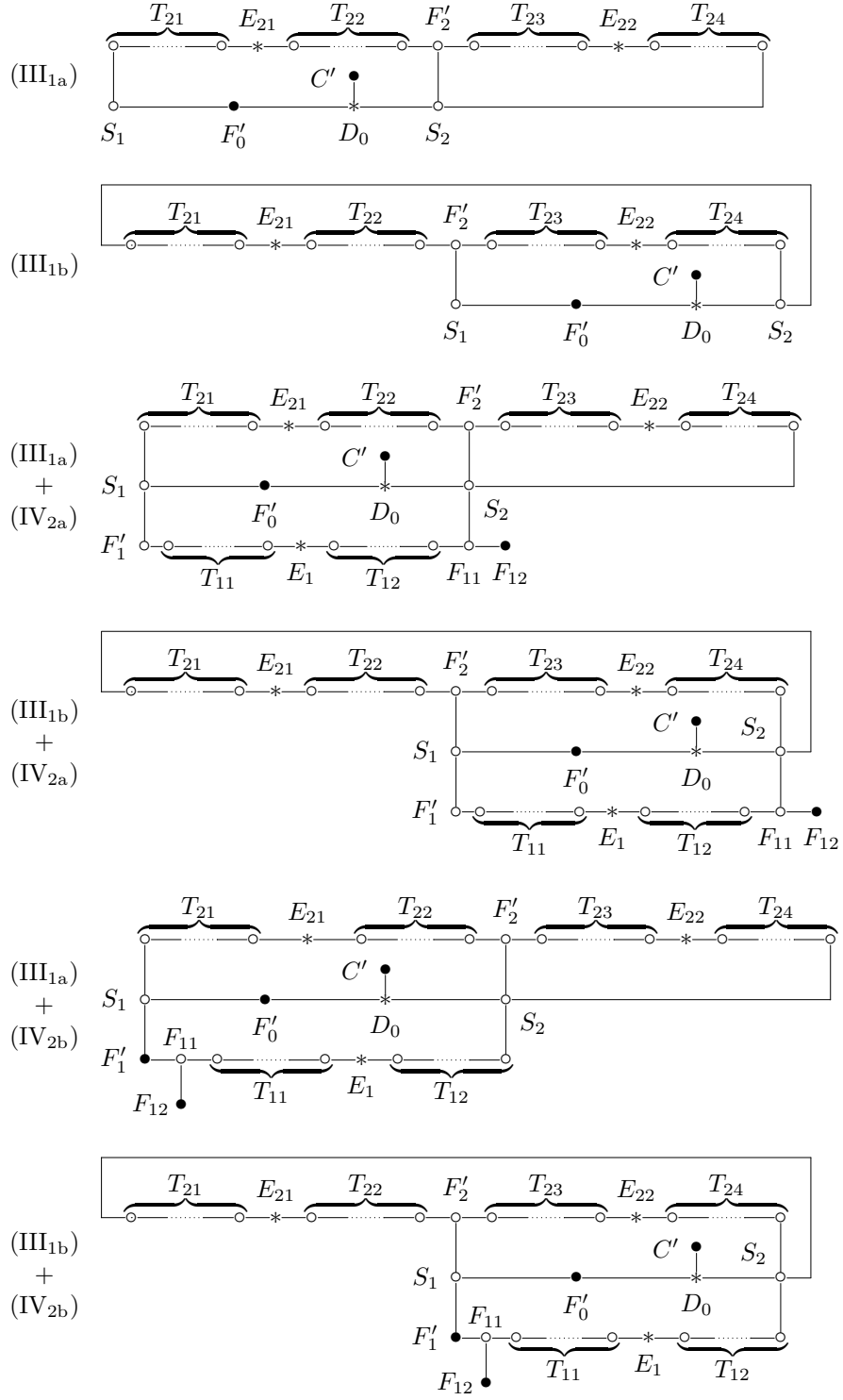


Figure 1: Dual graphs of $S_1 + S_2 + F_0 + F_1 + F_2$

(III_{1a}). We may assume $\varphi = \varphi_0 \circ \varphi_{21} \circ \varphi_{22}$, where φ_{22} (resp. φ_{21} , φ_0) contracts $T_{23} + E_{22} + T_{24}$ (resp. $\varphi_{22}(T_{21} + E_{21} + T_{22})$, $\varphi_{21}(\varphi_{22}(C' + D_0))$) to a point. We first show the following lemma.

LEMMA 20. *There exist positive integers k_{12} and k_{34} such that $[S_1, T_{21}]^* = [T_{22}, k_{12} + 1]$ and $[F'_2, T_{23}]^* = [T_{24}, k_{34} + 1, t_{k_{12}-1}]$. We have $k_{34} = s_2 + 2 \geq 5$, $T_{23} \neq \emptyset$, $B_g = [F'_0, S_1, T_{21}]$ and $A_g = t_{o_g} * [T_{22}, k_{12} + 2]$.*

PROOF. By Lemma 17, $\varphi_{21}(\varphi_{22}(S_1))$ is a (-1) -curve. The morphism φ_{22} does not change the linear chain $[S_1, T_{21}, E_{21}, T_{22}]$. We apply Corollary 9 to $[S_1, T_{21}, E_{21}, T_{22}]$ and φ_{21} . There exists a positive integer k_{12} such that $[S_1, T_{21}]^* = [T_{22}, k_{12} + 1]$. Since $\varphi_{21}(\varphi_{22}(F'_2))$ is a 0-curve, $\varphi_{22}(F'_2)$ must be a $(-k_{12})$ -curve by Corollary 10. Again by Corollary 9, there exists a positive integer k_{34} such that $[F'_2, T_{23}]^* = [T_{24}, k_{34} + 1, t_{k_{12}-1}]$. Since $\varphi(S_2)^2 = 4$, we have $4 = -s_2 + k_{34} + 2$ by Corollary 10. If $T_{23} = \emptyset$, then $[T_{24}, k_{34} + 1, t_{k_{12}-1}] = t_{f_2-1}$ by Lemma 5. We have $k_{34} = 1$. Thus $s_2 = -1$, which is absurd. Hence $T_{23} \neq \emptyset$. Either $A_g = {}^t[F'_0, S_1, T_{21}]$ or $B_g = [F'_0, S_1, T_{21}]$ by Lemma 12 (i). Suppose the former case holds. We have $g = 1$. Since $T_{23} \neq \emptyset$, we see $B_1 = [S_2, F'_2, T_{23}]$ and $T_{22} = \emptyset$. By Proposition 13 and Lemma 5, $[o_1 + 1, {}^tB_1] = {}^tA_1^* = [F'_0, S_1, T_{21}]^* = [S_1, T_{21}]^* * t_1 = [k_{12} + 2]$, which is a contradiction. Thus $B_g = [F'_0, S_1, T_{21}]$. By Proposition 13 and Lemma 5, $A_g = t_{o_g} * B_g^* = t_{o_g} * [S_1, T_{21}]^* * t_1 = t_{o_g} * [T_{22}, k_{12} + 2]$. \square

Case (i): $T_{24} = \emptyset$. By Lemma 20, $[F'_2, T_{23}] = [k_{34} + 1, t_{k_{12}-1}]^*$. By Lemma 5, $[k_{34} + 1, t_{k_{12}-1}]^* = [k_{12} + 1, t_{k_{34}-1}]$. Thus $f_2 = k_{12} + 1$ and $T_{23} = t_{k_{34}-1}$. Suppose $T_{22} \neq \emptyset$. We have $g = 2$ and $A_2 = [F'_2, S_2]$ by Lemma 12 (i). By Lemma 20, we obtain $o_2 = 1$, $[f_2 - 1] = T_{22}$, $s_2 = k_{12} + 2$ and $k_{34} = k_{12} + 4$. Either $T_{23} = {}^tA_1$ or $T_{23} = B_1$. Since T_{23} consists of (-2) -curves, it follows from Proposition 13 (iii) that $T_{23} = B_1$ and $T_{22} = A_1$. By Proposition 13, $T_{22} = A_1 = t_{o_1} * B_1^* = t_{o_1} * T_{23}^* = t_{o_1} * [k_{12} + 4]$. Thus $o_1 = 1$ and $f_2 = k_{12} + 6$, which contradicts $f_2 = k_{12} + 1$. Hence $T_{22} = \emptyset$. We have $g = 1$. By Lemma 20, $[S_1, T_{21}] = t_{k_{12}}$. This means that $A_1 = {}^t[S_2, F'_2, T_{23}]$ and $B_1 = [S_1, T_{21}]$. By Lemma 20, $[t_{k_{34}-1}, f_2, s_2] = t_{o_1} * [k_{12} + 2]$. We see $s_2 = k_{12} + 3$, $f_2 = 2$ and $o_1 = k_{34} + 1$. It follows that $k_{12} = 1$, $s_2 = 4$, $k_{34} = 6$ and $o_1 = 7$. We have $A_1 = [t_6, 4]$ and $[B_1, o_1 + 1] = A_1^* = [t_2, 8]$. The curve E_{22} intersects only the first and the last curve of A_1 among the irreducible components of D . By Lemma 15, C can be constructed as C_4 .

Case (ii): $T_{24} \neq \emptyset$. Since S_2 is a branching component of D , we infer $A_g = S_2$ by Lemma 12 (i). By Lemma 20, we obtain $o_g = 1$, $T_{22} = \emptyset$, $s_2 = k_{12} + 3$ and $k_{34} = k_{12} + 5$. We have $g = 2$. Either $B_1 = [F'_2, T_{23}]$ or $B_1 = {}^tT_{24}$. If $B_1 = [F'_2, T_{23}]$, then $T_{24} = A_1 = t_{o_1} * [F'_2, T_{23}]^* = t_{o_1} * [T_{24}, k_{34} + 1, t_{k_{12}-1}]$, which is impossible. Thus $B_1 = {}^tT_{24}$ and $A_1 = {}^t[F'_2, T_{23}]$. By Proposition 13, $[o_1 + 1, T_{24}] = {}^tA_1^* = [F'_2, T_{23}]^*$. By Lemma 20, $[o_1 + 1, T_{24}] = [T_{24}, k_{12} + 6, t_{k_{12}-1}]$. Hence $k_{12} = 1$, $[o_1 + 1, T_{24}] = [T_{24}, 7]$. It follows from Lemma 5 that $o_1 = 6$ and $T_{24} = [7_k]$, where $k = r(T_{24}) \geq 1$. We have $B_1 = [7_k]$,

$A_1 = t_{o_1} * B_1^* = t_6^{*k+1}$ and $A_2 = [4]$. Since $[B_2, o_2 + 1] = A_2^* = t_3$, we obtain $B_2 = t_2$. The curve E_{22} intersects only the first curve of A_1 and the last curve of B_1 among the irreducible components of D . By Lemma 15, C can be constructed as C_{4k+4} .

(III_{1a}) + (IV_{2a}). We have $A_g = S_2$ and $B_g = [F'_0, S_1, F'_1, T_{11}]$ because S_2 is a branching component of D . By Proposition 13, $[B_g, o_g + 1] = A_g^* = t_{s_2-1}$. Thus $[F'_1, T_{11}] = t_{s_2-4}$. By Lemma 19, φ contracts F_1 to a 0-curve, which is the image of F'_1 . By Lemma 4 (iii), $[T_{12}, F_{11}, F_{12}] = [F'_1, T_{11}]^* = t_{s_2-4}^* = [s_2 - 3]$, which is absurd. Hence this case does not occur.

(III_{1a}) + (IV_{2b}). We may assume $\varphi = \varphi_0 \circ \varphi_1 \circ \varphi_{21} \circ \varphi_{22}$, where φ_{22} (resp. φ_{21} , φ_1 , φ_0) contracts $T_{23} + E_{22} + T_{24}$ (resp. $\varphi_{22}(T_{21} + E_{21} + T_{22})$, $\varphi_{21}(\varphi_{22}(F_{11} + F_{12} + T_{11} + E_{11} + T_{12}))$, $\varphi_1(\varphi_{21}(\varphi_{22}(C' + D_0)))$ to a point. We show the following three lemmas.

LEMMA 21. *There exist positive integers k_{12} and k_{34} such that $[S_1, T_{21}]^* = [T_{22}, k_{12} + 1]$ and $[F'_2, T_{23}]^* = [T_{24}, k_{34} + 1, t_{k_{12}-1}]$. We have $[F_{11}, T_{11}]^* = [T_{12}, s_2 - k_{34} + 1]$ and $s_2 \geq k_{34} + 1$.*

PROOF. By the same arguments as in the proof of Lemma 20, there exist positive integers k_{12} , k_{34} such that $[S_1, T_{21}]^* = [T_{22}, k_{12} + 1]$ and $[F'_2, T_{23}]^* = [T_{24}, k_{34} + 1, t_{k_{12}-1}]$. By Lemma 19 and Corollary 9, there exists a positive integer l such that $[F_{11}, T_{11}]^* = [T_{12}, l + 1]$. Since $\varphi(S_2)^2 = 4$, we infer $4 = -s_2 + k_{34} + 2 + l + 2$. Thus $1 \leq l = s_2 - k_{34}$. \square

LEMMA 22. *We have $T_{21} = \emptyset$, $T_{22} = t_{s_1-2}$ and $k_{12} = 1$.*

PROOF. Suppose that S_1 is a branching component of D . We have $A_g = [S_1, F'_0]$, $T_{12} = T_{24} = \emptyset$ and $B_g = [S_2, F'_2, \dots]$. By Lemma 21, $[F_{11}, T_{11}] = t_{s_2-k_{34}}$ and $[F'_2, T_{23}] = [k_{12} + 1, t_{k_{34}-1}]$. By Proposition 13, $[B_g, o_g + 1] = A_g^* = t_1 * t_{s_1-1} = [3, t_{s_1-2}]$. Thus $o_g = 1$, $f_2 = 2$ and $s_2 = 3$. Since $f_2 = k_{12} + 1$, we obtain $k_{12} = 1$. Because $\emptyset \neq [F_{11}, T_{11}] = t_{3-k_{34}}$, we have $k_{34} \leq 2$. If $k_{34} = 1$, then $T_{23} = t_{k_{34}-1} = \emptyset$. Thus $B_g = [S_2, F'_2, {}^tT_{22}]$. By Proposition 13, $A_g = t_{o_g} * B_g^* = t_1 * [3, 2, {}^tT_{22}]^* = t_1 * [2, {}^tT_{22}]^* * t_2$. By Lemma 21, $t_1 * [2, {}^tT_{22}]^* * t_2 = t_1 * [{}^tT_{21}, S_1] * t_2$. This means that $S_1 = t_1 * [{}^tT_{21}, S_1] * t_1$, which is impossible. Hence $k_{34} = 2$. Since $T_{23} = [2] \neq \emptyset$, we infer $B_g = [S_2, F'_2, T_{23}]$ and $T_{22} = \emptyset$. By Lemma 21, $[S_1, T_{21}] = t_{k_{12}} = [2]$, which is absurd. Hence S_1 is not a branching component of D . We have $T_{21} = \emptyset$. By Lemma 21, $[T_{22}, k_{12} + 1] = t_{s_1-1}$. From this, we obtain $k_{12} = 1$ and $T_{22} = t_{s_1-2}$. \square

LEMMA 23. *We have $T_{11} = T_{12} = \emptyset$, $B_g = [F'_0, S_1, F'_1, F_{11}, F_{12}]$, $s_2 = k_{34} + 1$ and $F_{11} = [2]$.*

PROOF. Either $S_2 \subset A_g$ or $S_2 \subset B_g$. Suppose $S_2 \subset B_g$. We have $T_{24} = T_{12} = \emptyset$. By Lemma 21, $[F'_2, T_{23}] = [k_{34} + 1]^* = t_{k_{34}}$. Thus $f_2 = 2$, $T_{23} = t_{k_{34}-1}$. Since $[F_{11}, T_{11}] = t_{s_2-k_{34}}$, we get $F_{11} = [2]$ and $T_{11} =$

$t_{s_2-k_{34}-1}$. If $T_{11} \neq \emptyset$, then $A_1 = F_{12}$ or $A_1 = {}^tT_{11}$ since F_{11} is a branching component of D . Thus A_1 consists of (-2) -curves, which contradicts Proposition 13. Hence $T_{11} = \emptyset$. We have $s_2 = k_{34} + 1$, $g = 1$ and $A_1 = [F_{12}, F_{11}, F'_1, S_1, F'_0] = [t_3, S_1, 2]$. We infer $s_1 \geq 3$. By Proposition 13, $[B_1, o_1 + 1] = A_1^* = [3, t_{s_1-3}, 5]$. This means that $s_2 = 3$ and $k_{34} = 2$. Since $T_{23} = [2] \neq \emptyset$, we have $B_1 = [S_2, F'_2, T_{23}]$ and $T_{22} = \emptyset$. By Lemma 22, $s_1 = 2$, which is a contradiction. Hence $S_2 \subset A_g$. We have $B_g = [F'_0, S_1, F'_1, F_{11}, F_{12}]$ and $T_{11} = \emptyset$. By Lemma 21, $[T_{12}, s_2 - k_{34} + 1] = t_{-F_{11}^2-1}$. This shows $s_2 = k_{34} + 1$ and $T_{12} = t_{-F_{11}^2-2}$. If $T_{12} \neq \emptyset$, then $F_{11}^2 < -2$ and $A_g = S_2$. By Proposition 13, $[B_g, o_g + 1] = A_g^* = t_{s_2-1}$, which is absurd. Hence $T_{12} = \emptyset$ and $F_{11} = [2]$. \square

Case (i): $T_{24} = \emptyset$. By Lemma 21, $[F'_2, T_{23}] = t_{k_{34}}$. We have $f_2 = 2$ and $T_{23} = t_{k_{34}-1} = t_{s_2-2} \neq \emptyset$. If $T_{22} \neq \emptyset$, then $A_1 = T_{22}$ or $A_1 = {}^tT_{23}$. Thus A_1 consists of (-2) -curves, which contradicts Proposition 13. Hence $T_{22} = \emptyset$. We infer $g = 1$ and $A_1 = {}^t[S_2, F'_2, T_{23}] = [t_{k_{34}}, k_{34} + 1]$. By Lemma 22, we have $S_1 = [2]$ and $B_1 = t_5$. By Proposition 13, $A_1 = t_{o_1} * [6] = [t_{o_1-1}, 7]$. Hence $k_{34} = 6$, $A_1 = [t_6, 7]$. The curve E_{22} intersects only the first and the last curve of A_1 among the irreducible components of D . By Lemma 15, C can be constructed as C_4^* .

Case (ii): $T_{24} \neq \emptyset$. We have $A_g = S_2$. By Proposition 13, $[B_g, o_g + 1] = A_g^* = t_{s_2-1}$. We see $S_1 = [2]$, $B_g = t_5$, $s_2 = 7$ and $k_{34} = 6$ by Lemma 23. By Lemma 22, $T_{22} = \emptyset$. We infer $g = 2$. Either $B_1 = {}^tT_{24}$ or $A_1 = T_{24}$. If $A_1 = T_{24}$, then $B_1 = [F'_2, T_{23}]$. By Proposition 13 and Lemma 21, $T_{24} = t_{o_1} * [F'_2, T_{23}]^* = t_{o_1} * [T_{24}, 7]$, which is absurd. Hence $B_1 = {}^tT_{24}$ and $A_1 = {}^t[F'_2, T_{23}]$. By Proposition 13 and Lemma 21, $[o_1 + 1, T_{24}] = [F'_2, T_{23}]^* = [T_{24}, 7]$. It follows from Lemma 5 that $o_1 = 6$, $T_{24} = [7_k]$, where $k = r(T_{24}) \geq 1$. We have $B_2 = t_5$, $A_2 = [7]$, $B_1 = [7_k]$ and $A_1 = t_6^{*k+1}$. The curve E_{22} intersects only the first curve of A_1 and the last curve of B_1 among the irreducible components of D . By Lemma 15, C can be constructed as C_{4k+4}^* .

(III_{1b}), (III_{1b}) + (IV_{2a}) or (III_{1b}) + (IV_{2b}). In each case, we have $-2 \geq \varphi(S_1)^2$ because S_1 meets with only F'_i among the irreducible components of F_i for each i . Hence all the cases do not occur.

We list the weighted dual graphs of $D + E_1 + E_2$ in Figure 2, where $k = 0$ if $T_{24} = \emptyset$. We proved that if a rational unicuspidal plane curve C satisfies the conditions $(C')^2 = -2$, $\bar{\kappa} = 2$, then C can be constructed in the same way as C_{4k} or C_{4k}^* for some k . By Proposition 16, C is projectively equivalent to C_{4k} or C_{4k}^* . We have thus proved Theorem 1.

Finally, we prove Theorem 2. The “only if” part of Theorem 2 follows from [K2, Lemma 4.4]. We show the “if” part. By Theorem 3.1, Lemma 4.2, Lemma 4.5 and Lemma 4.6 of [K2], we deduce that if $\bar{\kappa}(\mathbf{P}^2 \setminus C) \geq 0$, $\bar{P}_2(\mathbf{P}^2 \setminus C) = \bar{P}_3(\mathbf{P}^2 \setminus C) = 0$, then C is a rational unicuspidal curve such that $\bar{\kappa} = 2$, $(C')^2 = -2$ and the dual graph of the exceptional curve of the

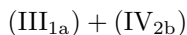


Figure 2: The dual graphs of $D + E_1 + E_2$

minimal embedded resolution of C is linear. Thus the “if” part follows from Theorem 1 and Lemma 15.

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Appendix by Fumio Sakai

Let N be the nodal cubic $x^3 + y^3 - xyz = 0$. Let O denote the node $(0, 0, 1)$. It is well known that the set $N \setminus \{O\}$ has a group structure, which is isomorphic to the multiplicative group \mathbf{C}^* . The group isomorphism is given by $\phi : \mathbf{C}^* \ni t \mapsto (t, -t^2, t^3 - 1) \in N \setminus \{O\}$. Geometrically, we have $t_1 t_2 t_3 = 1$ if and only if $\phi(t_1)$, $\phi(t_2)$ and $\phi(t_3)$ are collinear. We see easily that N has three flexes $O_1 = (1, -1, 0) = \phi(1)$, $O_2 = (1, -\omega, 0) = \phi(\omega)$ and $O_3 = (1, -\omega^2, 0) = \phi(\omega^2)$, where $\omega = e^{2\pi i/3}$. There exist three projective transformations

$$\varphi_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 & \omega^2 & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} 0 & \omega & 0 \\ \omega^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that $\varphi_i(O_i) = O_i$, $\varphi_i(O_j) = O_k$ for distinct i, j, k among $\{1, 2, 3\}$.

THEOREM 24. *Define three conics*

$$\begin{aligned} Q_1 &: 21(x^2 + y^2) - 22xy - 6(x + y)z + z^2 = 0, \\ Q_2 &: 21(\omega x^2 + \omega^2 y^2) - 22xy - 6(\omega^2 x + \omega y)z + z^2 = 0, \\ Q_3 &: 21(\omega^2 x^2 + \omega y^2) - 22xy - 6(\omega x + \omega^2 y)z + z^2 = 0. \end{aligned}$$

Then the conic Q_1 (resp. Q_2, Q_3) intersects N only at the point $P_1 = \phi(-1)$ (resp. $P_2 = \phi(-\omega), P_3 = \phi(-\omega^2)$).

Conversely, if Q is an irreducible conic with the property that Q intersects N only at a point $P \in N \setminus \{O\}$, then Q is one of the above three conics.

Note that the tangent line to Q_i at P_i passes through O_i for each i and that $\varphi_i(Q_i) = Q_i, \varphi_i(Q_j) = Q_k$ for distinct i, j, k among $\{1, 2, 3\}$.

PROOF. Let Q be a conic defined by the general equation:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0.$$

Suppose that Q intersects N only at a point $P = \phi(\alpha) \in N \setminus \{O\}$, where $\alpha \in \mathbf{C}^*$. Then we have

$$at^2 + bt^4 + c(t^3 - 1)^2 - dt^3 + et(t^3 - 1) - ft^2(t^3 - 1) = 0.$$

It follows that

$$ct^6 - ft^5 + (b + e)t^4 - (2c + d)t^3 + (a + f)t^2 - et + c = 0.$$

Since Q does not pass through O , we infer that $c \neq 0$. So we may assume that $c = 1$. Thus, we have

$$t^6 - ft^5 + (b + e)t^4 - (2 + d)t^3 + (a + f)t^2 - et + 1 = 0.$$

By our hypothesis, this equation must have only one multiple root α of order six. We see that $\alpha^6 = 1, f = 6\alpha, b + e = 15\alpha^2, 2 + d = 20\alpha^3, a + f = 15\alpha^4, e = 6\alpha^5$. In particular, α is a 6-th root of unity. We then obtain the equations of the conics Q_1, Q_2, Q_3 for $\alpha = -1, -\omega, -\omega^2$, respectively. For the cases in which $\alpha = 1, \omega, \omega^2$, the conic Q is reduced to a double tangent line at the flex O_1, O_2, O_3 , respectively. \square

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